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Some fixed point results for dual contractions of rational type

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Abstract. Isik and Turkoglu [Some Fixed Point Theorems in Ordered Partial Metric Spaces, Journal of Inequalities and Special Functions, Volume 4 Issue 2(2013), Pages 13-18] established new fixed point results in complete partial metric spaces.
The purpose of this paper is to introduce a dual contraction of rational type and to obtain some new fixed point results in dual partial metric spaces for dominating and dominated mappings. These results extend various comparable results existing in literature. Moreover, we give an example that shows the usefulness and effectiveness of these results among corresponding fixed point theorems established in partial metric spaces.

1. Introduction and preliminaries

In [7], Matthews introduced the concept of partial metric space as a suitable mathematical tool for program verification and proved an analogue of Banach fixed point theorem in complete partial metric spaces. Fixed point theorems in complete partial metric spaces have been investigated in [3, 4, 12] and references therein.
Neill [10] introduced the concept of dual partial metric (dual pmetric), which is more general than partial metric and established a robust relationship between dual partial metric and quasi metric. However, Oltra *et al.* [9] initiated the study of fixed points in dual pmetric and presented an analogue of Banach fixed point theorem. Recently, Nazam *et al.* [2, 8] established some fixed point results in dual pmetric spaces.
Harjani *et al.* [5] extended Banach's Contraction principle as follows:

Theorem 1.1. *Let M be a complete ordered metric space and $T : M \to M$ be a continuous and nondecreasing rational type contraction mapping. Then T has a unique fixed point $m^* \in M$. Moreover, the Picard iterative sequence $\{T^n(j)\}_{n\in\mathbb{N}}$ converges to m^* for every $j \in M$.*

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Isik and Tukroglu [6] established an ordered partial metric version of Theorem 1.1, stated below.

Theorem 1.2. *Let M be a complete ordered partial metric space and $T : M \to M$ be a continuous and nondecreasing rational type contraction mapping. Then T has a unique fixed point $m^* \in M$. Moreover, the Picard iterative sequence $\{T^n(j)\}_{n\in\mathbb{N}}$ converges to m^* for every $j \in M$.*

Throughout, in this paper, the letters $\mathbb{R}_0^+$, $\mathbb{R}$ and $\mathbb{N}$ will represent the set of nonnegative real numbers, set of real numbers and set of natural numbers, respectively.

Definition 1.1 ([1])**.** (1) Let M be a nonempty set and $T : M \to M$ be a self mapping. A point $m^* \in M$ is called a fixed point of T if $m^* = T(m^*)$.
(2) Let $(M, \preceq)$ be an ordered set and $T : M \to M$ be a self mapping defined on M satisfying the property $j \preceq T(j)$ for all $j \in M$. Then T is called a dominating mapping.
(3) Let $(M, \preceq)$ be an ordered set and $T : M \to M$ be a self mapping defined on M satisfying the property $T(j) \preceq j$ for all $j \in M$. Then T is called a dominated mapping.

Matthews pmetric is defined as

Definition 1.2. [7] Let M be a nonempty set and $p : M \times M \to \mathbb{R}_0^+$ satisfies following properties, for all $x, y, z \in M$

(p_1) $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$,
(p_2) $p(x,x) \leq p(x,y)$,
(p_3) $p(x,y) = p(y,x)$,
(p_4) $p(x,z) + p(y,y) \leq p(x,y) + p(y,z)$.

Then p is a pmetric.

Definition 1.3. Let M be a nonempty set and p be a partial metric on M. A functional $D : M \times M \to \mathbb{R}$ defined by

$$D(j,k) = p(j,k) - p(j,j) - p(k,k) \text{ for all } j,k \in M$$

satisfies $(p_1) - (p_4)$ that is

(D_1) $j = k \Leftrightarrow D(j,j) = D(k,k) = D(j,k)$.
(D_2) $D(j,j) \leq D(j,k)$.
(D_3) $D(j,k) = D(k,j)$.
(D_4) $D(j,l) \leq D(j,k) + D(k,l) - D(k,k)$.

called dual pmetric and the pair (M, D) is known as dual pmetric space.

Remark 1.1. We observe that, unlike pmetric case, if D is a dual pmetric then $D(j,k) = 0$ may not implies $j = k$. The self distance $D(j,k)$ referred to as the size or weight of j, is a feature used to describe the amount of information contained in j. The smaller $D(j,j)$ the more defined j is, j

being totally defined if $D(j,j) = 0$. It is obvious that if p is a partial metric then D is a dual partial metric but converse is not true. Note that $D(j,j) \leq D(j,k)$, does not imply $p(j,j) \leq p(j,k)$. Nevertheless, the restriction of D to $\mathbb{R}_0^+$, is a partial metric.

Definition 1.4 (Consistent Semilattice)**.** Let $(X,\preceq)$ be a poset such that

(1) for all $x, y \in X$ $x \wedge y \in X$,
(2) if $\{x,y\} \subseteq X$ is consistent, then $x \vee y \in X$,

then $(X,\preceq)$ with (1) and (2) is called consistent semilattice.

Definition 1.5 (Valuation Space)**.** A valuation space is a consistent semilattice $(X,\preceq)$ and a function $\mu : X \to \mathbb{R}$, called valuation, such that

(1) if $x \preceq y$ and $x \neq y$, $\mu(x) < \mu(y)$ and
(2) if $\{x,y\} \subseteq X$ is consistent, then

$$\mu(x) + \mu(y) = \mu(x \wedge y) + \mu(x \vee y).$$

Example 1.1. Suppose that $(M,\preceq,\mu)$ is a valuation space, then $D(j,k) = \mu(j \vee k)$ defines a dual pmetric on M.

Proof. Axioms (D_2) and (D_3) are immediate. For (D_1) we proceed as

$$\text{if } D(j,j) = D(j,k) = D(k,k), \\ \text{then } \mu(j \vee k) = \mu(j) = \mu(k) \text{ implies } j = k.$$

Converse is obvious.

We prove (D_4) in detail:

$$\begin{aligned} D(j,l) + D(k,k) &= \mu(j \vee l) + \mu(k) \\ &\leq \mu(j \vee k \vee l) + \mu[(j \vee k) \wedge (k \vee l)] \\ &= \mu(j \vee k \vee l) + \mu(j \vee k) + \mu(k \vee l) - \mu(j \vee k \vee l) \\ &= \mu(j \vee k) + \mu(k \vee l) \\ &= D(j,k) + D(k,l). \end{aligned}$$

□

Example 1.2. Let p be a pmetric defined on a non empty set $M = \{[a,b]; a \leq b\}$. The functional $D : M \times M \to \mathbb{R}$ defined by

$$D([a,b],[c,d]) = \begin{cases} c - d, & \text{if } \max\{b,d\} = b, \min\{a,c\} = a \\ a - b, & \text{if } \max\{b,d\} = d, \min\{a,c\} = c \end{cases}$$

defines a daul pmetric on M.

Example 1.3. Let d be a metric and p be a pmetric defined on a non empty set M and $c > 0$ be real number. The functional $D : M \times M \to \mathbb{R}$ defined by

$$D(j,k) = d(j,k) - c \quad \text{for all } j,k \in M$$

is a dual pmetric on M.

Following [10], each dual pmetric D on M generates a T_0 topology $\tau(D)$ on M. The elements of the topology $\tau(D)$ are open balls of the form $\{B_D(j,\epsilon) : j \in M, \epsilon > 0\}$, where $B_D(j,\epsilon) = \{k \in M : D(j,k) < \epsilon + D(j,j)\}$. A sequence $\{j_n\}_{n\in\mathbb{N}}$ in (M,D) converges to a point $j \in M$ if and only if $D(j,j) = \lim_{n\to\infty} D(j,j_n)$.

Definition 1.6 ([8])**.** Let (M,D) be a dual pmetric space.

(1) A sequence $\{j_n\}_{n\in\mathbb{N}}$ in (M,D) is called a Cauchy sequence if $\lim_{n,m\to\infty} D(j_n,j_m)$ exists and is finite.
(2) A dual pmetric space (M,D) is said to be complete if every Cauchy sequence $\{j_n\}_{n\in\mathbb{N}}$ in M converges, with respect to $\tau(D)$, to a point $j \in M$ such that $D(j,j) = \lim_{n,m\to\infty} D(j_n,j_m)$.

The following lemma will be helpful in the sequel.

Lemma 1.1 ([8, 9])**.** (1) *A dual pmetric (M,D) is complete if and only if the metric space (M,d^s_D) is complete.*
(2) *A sequence $\{j_n\}_{n\in\mathbb{N}}$ in M converges to a point $j \in M$ with respect to $\tau(d^s_D)$ if and only if*

$$\lim_{n\to\infty} D(j,j_n) = D(j,j) = \lim_{n\to\infty} D(j_n,j_m).$$

(3) *If $\lim_{n\to\infty} j_n = \upsilon$ such that $D(\upsilon,\upsilon) = 0$, then $\lim_{n\to\infty} D(j_n,k) = D(\upsilon,k)$ for every $k \in M$.*

Motivated by the results presented by Isik and Tukroglu [6] and Valero [9], we present a new fixed point theorem in an ordered dual pmetric space for both dominating and dominated mappings.

2. MAIN RESULTS

We begin with the following definition.

Definition 2.1. Let $(M,\preceq,D)$ be a complete ordered dual pmetric space. A mapping $T : M \to M$ is said to be a dual contraction of rational type if there exist nonnegative functions α,β satisfying

$$\sup_{j,k\in M} \{\alpha(j,k) + \beta(j,k)\} \leq \lambda < 1$$

such that for all comparable $j,k \in M$,

$$|D(T(j),T(k))| \leq \alpha(j,k) \left| \frac{D(j,T(j)) \cdot D(k,T(k))}{D(j,k)} \right| + \beta(j,k)|D(j,k)| \tag{1}$$

Theorem 2.1. *Let $(M,\preceq,D)$ be a complete ordered dual pmetric space. Suppose that $T : M \to M$ is a mapping such that*

(1) *T is a dual contraction of rational type;*
(2) *T is a dominating mapping;*
(3) *T is a continuous mapping.*

Then T has a fixed point m^*. Moreover, $D(m^*, m^*) = 0$.

Proof. Let j_0 be an initial point in M and $j_n = T(j_{n-1})$ be an iterative sequence. If there exists a positive integer r such that $j_{r+1} = j_r$, then j_r is the fixed point of T and it completes the proof. If $j_n \neq j_{n+1}$ for any $n \in \mathbb{N}$, then since T is dominating, $j_n \preceq T(j_n)$ for all $n \in \mathbb{N}$. Therefore,

$$j_0 \preceq j_1 \preceq j_2 \preceq j_3 \preceq \cdots \preceq j_n \preceq j_{n+1} \cdots .$$

Now since $j_n \preceq j_{n+1}$, by (1), we have

$$\begin{aligned} |D(j_n, j_{n+1})| &= |D(T(j_{n-1}), T(j_n))| \\ &\leq \alpha(j_{n-1}, j_n) \left| \frac{D(j_{n-1}, j_n) \cdot D(j_n, j_{n+1})}{D(j_{n-1}, j_n)} \right| \\ &\quad + \beta(j_{n-1}, j_n)|D(j_{n-1}, j_n)| \\ &\leq \alpha(j_{n-1}, j_n)|D(j_n, j_{n+1})| + \beta(j_{n-1}, j_n)|D(j_{n-1}, j_n)|, \end{aligned}$$

$$\begin{aligned} |D(j_n, j_{n+1})| - \alpha(j_{n-1}, j_n)|D(j_n, j_{n+1})| &\leq \beta(j_{n-1}, j_n)|D(j_{n-1}, j_n)|, \\ (1 - \alpha(j_{n-1}, j_n))|D(j_n, j_{n+1})| &\leq \beta(j_{n-1}, j_n)|D(j_{n-1}, j_n)|, \\ |D(j_n, j_{n+1})| &\leq \frac{\beta(j_{n-1}, j_n)}{1 - \alpha(j_{n-1}, j_n)}|D(j_{n-1}, j_n)|. \end{aligned}$$

If $\gamma = \dfrac{\beta(j_{n-1}, j_n)}{1 - \alpha(j_{n-1}, j_n)}$, then $0 < \gamma < 1$ and we have

$$|D(j_n, j_{n+1})| \leq \gamma|D(j_{n-1}, j_n)|.$$

Thus

$$|D(j_n, j_{n+1})| \leq \gamma|D(j_{n-1}, j_n)| \leq \gamma^2|D(j_{n-2}, j_{n-1})| \leq \cdots \leq \gamma^n|D(j_0, j_1)|.$$

Since $j_n \preceq j_n$, for each $n \in \mathbb{N}$, by (1), we have

$$|D(j_n, j_n)| \leq \frac{\alpha(j_{n-1}, j_{n-1})|D(j_{n-1}, j_n)|^2}{|D(j_{n-1}, j_{n-1})|} + \beta(j_{n-1}, j_{n-1})|D(j_{n-1}, j_{n-1})|,$$

implies

$$|D(j_n, j_n)| \leq |D(j_{n-1}, j_{n-1})| \left\{ \frac{\alpha(j_{n-1}, j_{n-1})|D(j_{n-1}, j_n)|^2}{|D(j_{n-1}, j_{n-1})|^2} + \beta(j_{n-1}, j_{n-1}) \right\},$$

so,

$$|D(j_n, j_n)| \leq (\alpha(j_{n-1}, j_{n-1}) + \beta(j_{n-1}, j_{n-1}))|D(j_{n-1}, j_{n-1})|.$$

Indeed, $\left| \dfrac{D(j_{n-1}, j_n)}{D(j_{n-1}, j_{n-1})} \right|^2 = 1$. Thus we obtain that

$$(2) \quad |D(j_n, j_n)| \leq \delta^n |D(j_0, j_0)|, \text{ where } \delta = \alpha(j_{n-1}, j_{n-1}) + \beta(j_{n-1}, j_{n-1})$$

Let show that $\{j_n\}$ is a Cauchy sequence in (M, d_D^s). Note that $d_D(j_n, j_{n+1}) = D(j_n, j_{n+1}) - D(j_n, j_n)$, that is, $d_D(j_n, j_{n+1}) + D(j_n, j_n) = D(j_n, j_{n+1}) \leq |D(j_n, j_{n+1})|$.

Thus we have

$$d_D(j_n, j_{n+1}) + D(j_n, j_n) \leq \gamma^n |D(j_0, j_1)|,$$
$$d_D(j_n, j_{n+1}) \leq \gamma^n |D(j_0, j_1)| + |D(j_n, j_n)|$$
$$\leq \gamma^n |D(j_0, j_1)| + \delta^n |D(j_0, j_0)|.$$

Continuing this way, we obtain that

$$d_D(j_{n+k-1}, j_{n+k}) \leq \gamma^{n+k-1} |D(j_0, j_1)| + \delta^{n+k-1} |D(j_0, j_0)|.$$

Now

$$\begin{aligned} d_D(j_n, j_{n+k}) &\leq d_D(j_n, j_{n+1}) + d_D(j_{n+1}, j_{n+2}) + \cdots + d_D(j_{n+k-1}, j_{n+k}) \\ &\leq \{\gamma^n + \gamma^{n+1} + \cdots + \gamma^{n+k-1}\} |D(j_0, j_1)| \\ &\quad + \{\delta^n + \delta^{n+1} + \cdots + \delta^{n+k-1}\} |D(j_0, j_0)|. \end{aligned}$$

Thus, for $n + k = m > n$,

$$d_D(j_n, j_m) \leq \frac{\gamma^n}{1-\gamma} |D(j_0, j_1)| + \frac{\delta^n}{1-\delta} |D(j_0, j_0)|.$$

Similarly, we have

$$d_D(j_m, j_n) \leq \frac{\gamma^n}{1-\gamma} |D(j_1, j_0)| + \frac{\delta^n}{1-\delta} |D(j_0, j_0)|.$$

Taking limit as $n, m \to \infty$, we have

$$\lim_{n,m\to\infty} d_D(j_m, j_n) = 0 = \lim_{n,m\to\infty} d_D(j_n, j_m) \text{ and hence } \lim_{n,m\to\infty} d_D^s(j_m, j_n) = 0.$$

Thus $\{j_n\}$ is a Cauchy sequence in (M, d_D^s). Since (M, D) is a complete dualistic partial metric space, by Lemma 1.1, the metric space (M, d_D^s) is also complete. Therefore, there exists $m^* \in M$ such that $\lim_{n\to\infty} d_D^s(j_n, m^*) = 0$. Again from Lemma 1.1, we have

$$\lim_{n\to\infty} d_D^s(j_n, m^*) = 0 \iff D(m^*, m^*) = \lim_{n\to\infty} D(j_n, m^*) = \lim_{n,m\to\infty} D(j_m, j_n).$$

Now $\lim_{n,m\to\infty} d_D(j_m, j_n) = 0$ implies $\lim_{n,m\to\infty} [D(j_m, j_n) - D(j_n, j_n)] = 0$ and hence

$$\lim_{n,m\to\infty} D(j_n, j_m) = \lim_{n\to\infty} D(j_n, j_n).$$

By (2), we have $\lim_{n\to\infty} D(j_n, j_n) = 0$. Consequently, $\lim_{n,m\to\infty} D(j_n, j_m) = 0$ and $\{j_n\}$ is a Cauchy sequence in (M, D). Thus

$$D(m^*, m^*) = \lim_{n\to\infty} D(j_n, m^*) = 0. \tag{3}$$

Now $d_D(m^*, T(m^*)) = D(m^*, T(m^*)) - D(m^*, m^*) = D(m^*, T(m^*))$ implies $D(m^*, T(m^*)) \geq 0$. Since T is continuous, for a given $\epsilon > 0$, there exists a $\delta > 0$ such that $T(B_D(m^*, \delta)) \subseteq B_D(T(m^*), \epsilon)$.

Since $\lim_{n\to\infty} D(j_{n+1}, m^*) = D(m^*, m^*) = 0$, there exists an $r \in \mathbb{N}$ such that $D(j_n, m^*) < D(m^*, m^*) + \delta$ for all $n \geq r$, and so $\{j_n\} \subset B_D(m^*, \delta)$ for all $n \geq r$. This implies that $T(j_n) \in T(B_D(m^*, \delta) \subseteq B_D(T(m^*), \epsilon)$ and so $D(T(j_n), T(m^*)) < D(T(m^*), T(m^*)) + \epsilon$ for all $n \geq r$.

Now for any $\epsilon > 0$, we know that

$$-\epsilon + D(T(m^*), T(m^*)) < D(T(m^*), T(m^*)) \leq D(j_{n+1}, T(m^*)),$$

which implies

$$|D(j_{n+1}, T(m^*)) - D(T(m^*), T(m^*))| < \epsilon.$$

That is, $D(T(m^*), T(m^*)) = \lim_{n\to\infty} D(j_{n+1}, T(m^*))$. The uniqueness of limit in $\mathbb{R}$ implies

$$\lim_{n\to\infty} D(j_{n+1}, T(m^*)) = D(T(m^*), T(m^*)) = D(m^*, T(m^*)). \tag{4}$$

Finally, we have

$$D(T(m^*), m^*) = \lim_{n\to\infty} D(T(j_n), j_n) = \lim_{n\to\infty} D(j_{n+1}, j_n) = 0.$$

This shows that $D(m^*, T(m^*)) = 0$. So from (3) and (4) we deduce that

$$D(m^*, T(m^*)) = D(T(m^*), T(m^*)) = D(m^*, m^*).$$

This leads us to conclude that $m^* = T(m^*)$ and hence m^* is a fixed point of T. □

In order to prove the uniqueness of fixed point of a mapping T in the above theorem, we give the following theorem.

Theorem 2.2. *Let $(M, \preceq, D)$ be a complete ordered dual pmetric space and $T : M \to M$ be a mapping which satisfies all the conditions of Theorem 2.1. Then T has a unique fixed point provided that for each fixed point m^*, n^* of T, there exists $\omega \in M$ which is comparable to both m^* and n^*.*

Proof. From Theorem 2.1, it follows that the set of fixed points of T is nonempty.

Assume that n^* is another fixed point of T, that is, $n^* = T(n^*)$.

Case I. m^* and n^* are comparable.
In this case, we have

$$\begin{aligned}|D(m^*, n^*)| &= |D(T(m^*), T(n^*))| \\ &\leq \alpha(m^*, n^*)\left|\frac{D(m^*, T(m^*)) \cdot D(n^*, T(n^*))}{D(m^*, n^*)}\right| \\ &\quad + \beta(m^*, n^*)|D(m^*, n^*)| \\ &\leq \alpha(m^*, n^*)\left|\frac{D(m^*, m^*) \cdot D(n^*, n^*)}{D(m^*, n^*)}\right| \\ &\quad + \beta(m^*, n^*)|D(m^*, n^*)|.\end{aligned}$$

That is, $(1-\beta(m^*,n^*))|D(m^*,n^*)| \leq 0$ which implies that $|D(m^*,n^*)| \leq 0$ and hence $D(m^*,n^*) = 0 = D(m^*,m^*) = D(n^*,n^*)$. So m^* is a unique fixed point of T.

Case II. m^* and n^* are incomparable.

In this case, there exists ω which is comparable to both m^*, n^*. Without any loss of generality, we assume that $m^* \preceq \omega$ and $n^* \preceq \omega$. Since T is dominating, $m^* \preceq T(\omega)$ and $n^* \preceq T(\omega)$ imply that $m^* \preceq T^{n-1}(\omega)$ and $n^* \preceq T^{n-1}(\omega)$. Thus

$$\begin{aligned}&|D(m^*,T^n(\omega))|\\&\leq \alpha(n^*,T^{n-1}(\omega))\left|\frac{D(T^{n-1}(m^*),T^n(m^*))\cdot D(T^{n-1}(\omega),T^n(\omega))}{D(T^{n-1}(m^*),T^{n-1}(\omega))}\right|\\&\quad+\beta(n^*,T^{n-1}(\omega))|D(T^{n-1}(m^*),T^{n-1}(\omega))|,\end{aligned}$$

which implies that $|D(m^*,T^n(\omega))| \leq \beta(n^*,T^{n-1}(\omega))|D(m^*,T^{n-1}(\omega))|$. Thus $\lim_{n\to\infty} D(m^*,T^n(\omega)) = 0$.
Similarly, $\lim_{n\to\infty} D(n^*,T^n(\omega)) = 0$. Moreover, by D_4,

$$\begin{aligned}D(n^*,m^*) &\leq D(n^*,T^n(\omega)) + D(T^n(\omega),m^*) - D(T^n(\omega),T^n(\omega))\\&\leq D(n^*,T^n(\omega)) + D(T^n(\omega),m^*) - D(T^n(\omega),m^*)\\&\quad - D(m^*,T^n(\omega)) + D(m^*,m^*).\end{aligned}$$

Letting $n\to\infty$, we obtain that $D(n^*,m^*) \leq 0$ but $d_D(m^*,m^*) = D(n^*,m^*) - D(n^*,n^*)$ implies that $D(n^*,m^*) \geq 0$. Hence $D(n^*,m^*) = 0$ which gives that $n^* = m^*$. □

Remark 2.1. Since every partial metric is a dual pmetric D with $D(j,k) \in \mathbb{R}_0^+$ for all $j,k \in M$, Theorem 2.1 is an extension of Theorem 1.2.There arises the following natural question:
Whether the contractive condition (1) in the statement of Theorem 2.1 can be replaced by the contractive condition in Theorem 1.2?

The following example provides a negative answer to the above question.

Example 2.1. Define the mapping $T_0 : \mathbb{R} \to \mathbb{R}$ by

$$T_0(j) = \begin{cases} 0, & \text{if } j > 1,\\ -\dfrac{1}{2}, & \text{if } j = -5.\end{cases}$$

Clearly, for any $j,k \in \mathbb{R}$ and $\alpha(j,k) = 0.1$ and $\beta(j,k) = 0.09$, the following contractive condition is satisfied

$$D_\vee(T_0(j),T_0(k)) \leq \frac{\alpha(j,k)D_\vee(j,T_0(j))\cdot D_\vee(k,T_0(k))}{D_\vee(j,k)} + \beta(j,k)D_\vee(j,k),$$

where $D_\vee$ is a complete dual pmetric on $\mathbb{R}$. Here T has no fixed point. Thus a fixed point free mapping satisfies the contractive condition of Theorem

1.2. On the other hand, for $\alpha(j,k) = 0.1$ and $\beta(j,k) = 0.09$, we have

$$0.5 = |D_\vee(-\frac{1}{2}, -\frac{1}{2})| = |D_\vee(T_0(-5), T_0(-5))|$$

$$> 0.455 = \left|\frac{\alpha D_\vee(-5, T_0(-5)) \cdot D(-5, T_0(-5))}{D_\vee(-5,-5)}\right| + \beta|D_\vee(-5,-5)|.$$

Thus the contractive condition (1) of Theorem 2.1 does not hold.

In next theorem, we show that the conclusion of Theorem 2.1 remains the same if the continuity of the mapping T is replaced with the following condition:

(H): If $\{j_n\}$ is a nondecreasing sequence in M such that $j_n \to \upsilon$, then $j_n \preceq \upsilon$ for all $n \in \mathbb{N}$.

For dominated mappings, the following condition will be needed:

(Q): If $\{j_n\}$ is a nonincreasing sequence in M such that $j_n \to \upsilon$, then $j_n \succeq \upsilon$ for all $n \in \mathbb{N}$.

Theorem 2.3. *Let $(M, \preceq, D)$ be a complete ordered dual pmetric space. Suppose that $T : M \to M$ is a mapping such that*

(1) *T is a dual contraction of rational type;*
(2) *T is a dominating mapping;*
(3) (H) *holds.*

Then T has a fixed point m^*. Moreover, $D(m^*, m^*) = 0$.

Proof. By the arguments similar to those in the proof of Theorem 2.1, we obtain that $\{j_n\}$ is a nondecreasing sequence in M such that $j_n \to m^*$. By (H), we have $j_n \preceq m^*$. Since T is a dominating mapping, we have $j_n \preceq T(m^*)$ and

$$m^* \preceq T(m^*). \tag{5}$$

From the proof of Theorem 2.1, we deduce that $\{T^n(m^*)\}$ is a nondecreasing sequence. Suppose that $\lim_{n\to+\infty} T^n(m^*) = \mu$ for some $\mu \in M$. Now $j_n \preceq m^*$ implies $j_n \preceq T^n(m^*)$ for all $n \geq 1$. Thus we have

$$j_n \preceq m^* \preceq T(m^*) \preceq T^n(m^*) \quad n \geq 1.$$

From (1), we have

$$\begin{aligned} |D(j_{n+1}, T^{n+1}(m^*))| &= |D(T(j_n), T(T^n(m^*)))| \\ &\leq \left|\frac{\alpha(j_n, T^n(m^*)) D(j_n, j_{n+1}) \cdot D(T^n(m^*), T^{n+1}(m^*))}{D(j_n, T^n(m^*))}\right| \\ &\quad + \beta(j_n, T^n(m^*))|D(j_n, T^n(m^*))|. \end{aligned}$$

Taking limit as $n \to +\infty$, we obtain that

$$|D(m^*, \mu)| \leq \beta |D(m^*, \mu)|,$$

which implies that $m^* = \mu$. Thus $\lim_{n\to+\infty} T^n(m^*) = \mu$ implies that $\lim_{n\to+\infty} T^n(m^*) = m^*$. Hence

$$T(m^*) \preceq m^*. \tag{6}$$

From (5) and (6), it follows that $m^* = T(m^*)$. □

Now we present some important consequences of our results.

Corollary 2.1. *Let $(M, \preceq, D)$ be a complete ordered dual pmetric space. Suppose that $T : M \to M$ is a mapping such that*

(1) *T is a dual contraction of rational type with $\beta(j,k) = 0$;*
(2) *T is a dominating mapping;*
(3) *T is a continuous mapping.*

Then T has a fixed point m^*. Moreover, $D(m^*, m^*) = 0$.

Corollary 2.2. *Let $(M, \preceq, D)$ be a complete ordered dual pmetric space. Suppose that $T : M \to M$ is a mapping such that*

(1) *T is a dual contraction of rational type with $\beta(j,k) = 0$;*
(2) *T is a dominating mapping;*
(3) (H) *holds.*

Then T has a fixed point m^*. Moreover, $D(m^*, m^*) = 0$.

Corollary 2.3. *Let $(M, \preceq, D)$ be a complete ordered dual pmetric space. Suppose that $T : M \to M$ is a mapping such that*

(1) *T is a dual contraction of rational type with $\alpha(j,k) = 0$;*
(2) *T is a dominating mapping;*
(3) *T is a continuous mapping.*

Then T has a fixed point m^*. Moreover, $D(m^*, m^*) = 0$.

Corollary 2.4. *Let $(M, \preceq, D)$ be a complete ordered dual pmetric space. Suppose that $T : M \to M$ is a mapping such that*

(1) *T is a dual contraction of rational type with $\alpha(j,k) = 0$;*
(2) *T is a dominating mapping;*
(3) (H) *holds.*

Then T has a fixed point m^*. Moreover, $D(m^*, m^*) = 0$.

For dominated mappings, we present the following results.

Theorem 2.4. *Let $(M, \preceq, D)$ be a complete ordered dual pmetric space. Suppose that $T : M \to M$ is a mapping such that*

(1) *T is a dual contraction of rational type;*
(2) *T is a dominated mapping;*
(3) *T is a continuous mapping.*

Then T has a fixed point m^*. Moreover, $D(m^*, m^*) = 0$.

Proof. Let $j_0 \in M$ be an initial element and $j_n = T(j_{n-1})$ for all $n \geq 1$. If there exists a positive integer r such that $j_{r+1} = j_r$ then $j_r = T(j_r)$, and so we are done. Suppose that $j_n \neq j_{n+1}$ for all $n \in \mathbb{N}$. Since T is a dominated mapping, $j_0 \succeq T(j_0) = j_1$, and $j_1 \succeq T(j_1)$ implies $j_1 \succeq j_2$, and $j_2 \succeq T(j_2)$ implies $j_2 \succeq j_3$. Continuing in the similar way, we get

$$j_0 \succeq j_1 \succeq j_2 \succeq j_3 \succeq \cdots \succeq j_n \succeq j_{n+1} \succeq j_{n+2} \succeq \cdots .$$

The remaining part of the proof follows from the proof of Theorem 2.1. □

The following example shall illustrate Theorem 2.4.

Example 2.2. Let $M = \mathbb{R}^2$. Define $D_\vee : M \times M \to \mathbb{R}$ by $D_\vee(j,k) = j_1 \vee k_1$, where $j = (j_1, j_2)$ and $k = (k_1, k_2)$. Note that $(M, D_\vee)$ is a complete dual partial metric space. Let $T : M \to M$ be given by

$$T(j) = \frac{j}{2} \quad \text{for all } j \in M.$$

In M, we define the relation $\succeq$ in the following way:

$$j \succeq k \text{ if and only if } j_1 \geq k_1, \text{ where } j = (j_1, j_2) \text{ and } k = (k_1, k_2).$$

Clearly, $\succeq$ is a partial order on M and T is a continuous, dominated mapping with respect to $\succeq$. Moreover, $T(-1,0) \succeq (-1,0)$. We shall show that for all $j, k \in M$, the contractive condition (1) is satisfied. For this, note that

$$|D_\vee(T(j), T(k))| = \left| D_\vee\left(\frac{j}{2}, \frac{k}{2}\right)\right| = \left|\frac{j_1}{2}\right| \quad \text{for all } j_1 \geq k_1,$$

$$|D_\vee(j, T(j))| = \left| D_\vee\left(j, \frac{j}{2}\right)\right| = \begin{cases} \left|\frac{j_1}{2}\right|, & \text{if } j_1 \leq 0, \\ |j_1|, & \text{if } j_1 \geq 0, \end{cases}$$

$$|D_\vee(k, T(k))| = \left| D_\vee\left(k, \frac{k}{2}\right)\right| = \begin{cases} \left|\frac{k_1}{2}\right|, & \text{if } k_1 \leq 0, \\ |k_1|, & \text{if } k_1 \geq 0, \end{cases}$$

$$|D_\vee(j,k)| = |j_1|, \quad \text{for all } j_1 \geq k_1.$$

Now for $\alpha = \frac{1}{3}$, $\beta = \frac{1}{2}$. If $j_1 \leq 0$, $k_1 \leq 0$, then

$$|D_\vee(T(j), T(k))| \leq \alpha(j,k) \left| \frac{D_\vee(j, T(j)) \cdot D_\vee(k, T(k))}{D_\vee(j,k)} \right| + \beta(j,k) |D_\vee(j,k)|$$

holds for all $j \succeq k$ if and only if $6|j_1| \leq |k_1| + 6|j_1|$.

If $j_1 \geq 0$, $k_1 \geq 0$, then the contractive condition

$$|D_\vee(T(j), T(k))| \leq \alpha(j,k) \left| \frac{D_\vee(j, T(j)) \cdot D_\vee(k, T(k))}{D_\vee(j,k)} \right| + \beta(j,k) |D_\vee(j,k)|$$

holds for all $j \succeq k$ if and only if $j_1 \leq \frac{2}{3}k_1 + j_1$.

Finally, if $j_1 \geq 0$, $k_1 \leq 0$, then

$$|D_\vee(T(j), T(k))| \leq \alpha(j,k) \left| \frac{D_\vee(j, T(j)) \cdot D_\vee(k, T(k))}{D_\vee(j,k)} \right| + \beta(j,k)|D_\vee(j,k)|$$

holds for all $j \succeq k$ if and only if $3j_1 \leq |k_1| + 3j_1$. Thus all the conditions of Theorem 2.3 are satisfied. Moreover, $(0,0)$ is a fixed point of T.

Theorem 2.5. *Let $(M, \preceq, D)$ be a complete ordered dual pmetric space. Suppose that $T : M \to M$ is a mapping such that*

(1) *T is a dual contraction of rational type;*
(2) *T is a dominated mapping;*
(3) (Q) *holds.*

Then T has a fixed point m^*. Moreover, $D(m^*, m^*) = 0$.

The proof can be obtained by the proofs of Theorems 2.3 and 2.4. Some consequences of Theorems 2.4 and 2.5 are given below.

Corollary 2.5. *Let $(M, \preceq, D)$ be a complete ordered dual pmetric space. Suppose that $T : M \to M$ is a mapping such that*

(1) *T is a dual contraction of rational type with $\beta(j,k) = 0$;*
(2) *T is a dominated mapping;*
(3) *T is a continuous mapping.*

Then T has a fixed point m^*. Moreover, $D(m^*, m^*) = 0$.

Corollary 2.6. *Let $(M, \preceq, D)$ be a complete ordered dual pmetric space. Suppose that $T : M \to M$ is a mapping such that*

(1) *T is a dual contraction of rational type with $\beta(j,k) = 0$;*
(2) *T is a dominated mapping;*
(3) (Q) *holds.*

Then T has a fixed point m^*. Moreover, $D(m^*, m^*) = 0$.

Corollary 2.7. *Let $(M, \preceq, D)$ be a complete ordered dual pmetric space. Suppose that $T : M \to M$ is a mapping such that*

(1) *T is a dual contraction of rational type with $\alpha(j,k) = 0$;*
(2) *T is a dominated mapping;*
(3) *T is a continuous mapping.*

Then T has a fixed point m^*. Moreover, $D(m^*, m^*) = 0$.

Corollary 2.8. *Let $(M, \preceq, D)$ be a complete ordered dual pmetric space. Suppose that $T : M \to M$ is a mapping such that*

(1) *T is a dual contraction of rational type with $\alpha(j,k) = 0$;*
(2) *T is a dominated mapping;*
(3) (Q) *holds.*

Then T has a fixed point m^*. Moreover, $D(m^*, m^*) = 0$.

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